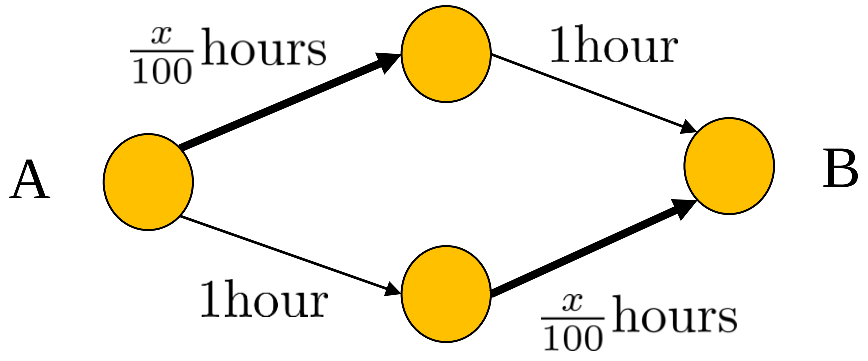
L13 Price of Anarchy

CS 280 Algorithmic Game Theory

Ioannis Panageas

Price of Anarchy

Suppose 100 drivers commute from A to B.
Drivers want to minimize the time.



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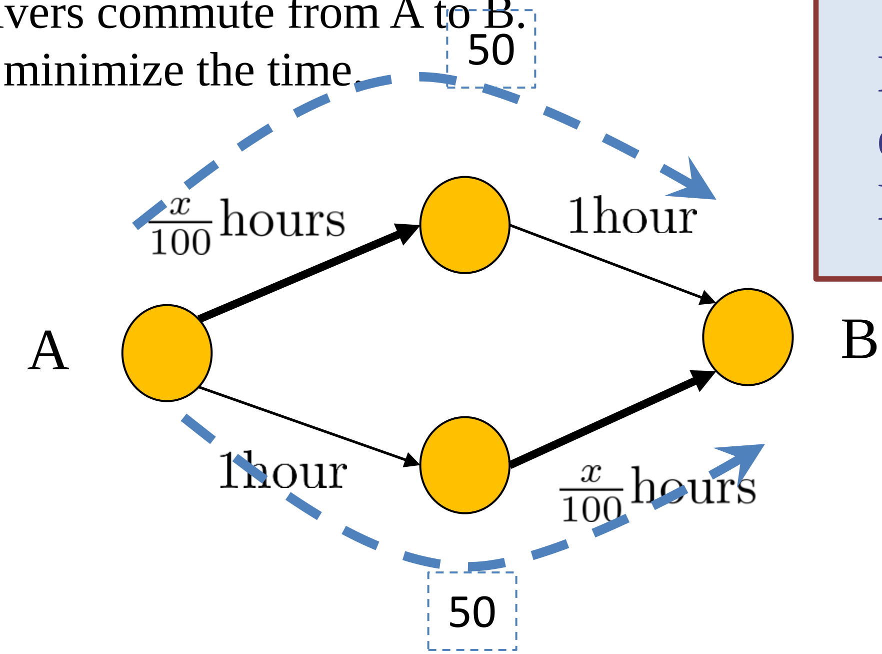

Delay is 1.5 hours for everybody at the unique Nash equilibrium.

Price of Anarchy

Suppose 100 drivers commute from A to B.
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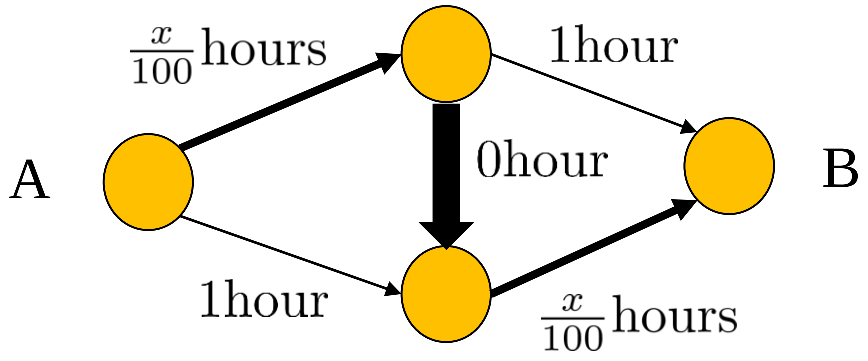
Question: What if we add a new link?



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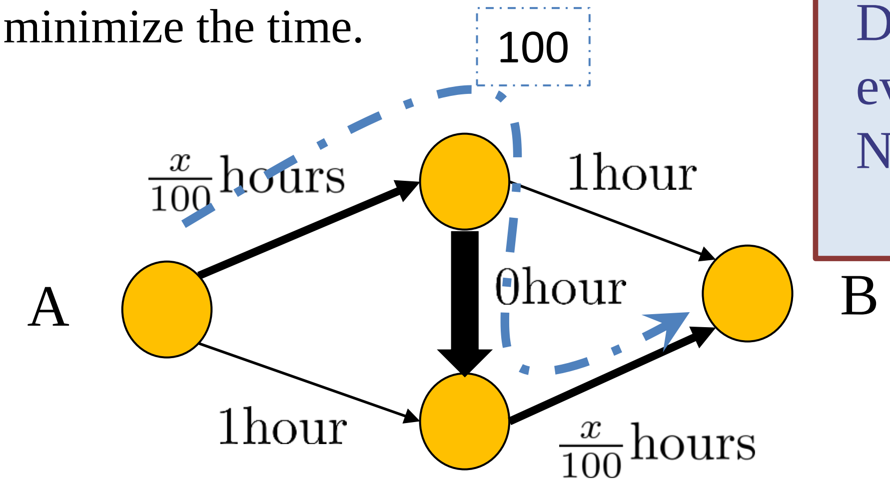

Delay is now 2 hours for everybody at the unique Nash equilibrium.
Braess's paradox

Adding a fast link is not always a good idea!

Price of Anarchy

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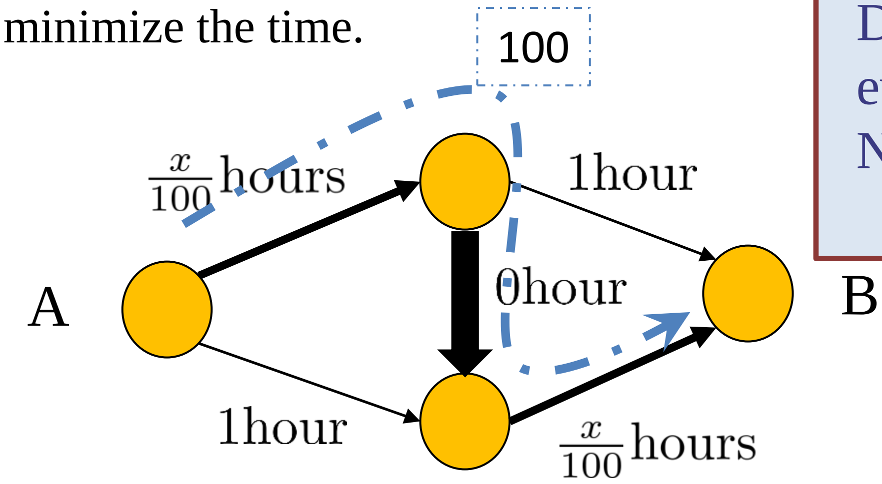

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Adding a fast link is not always a good idea!

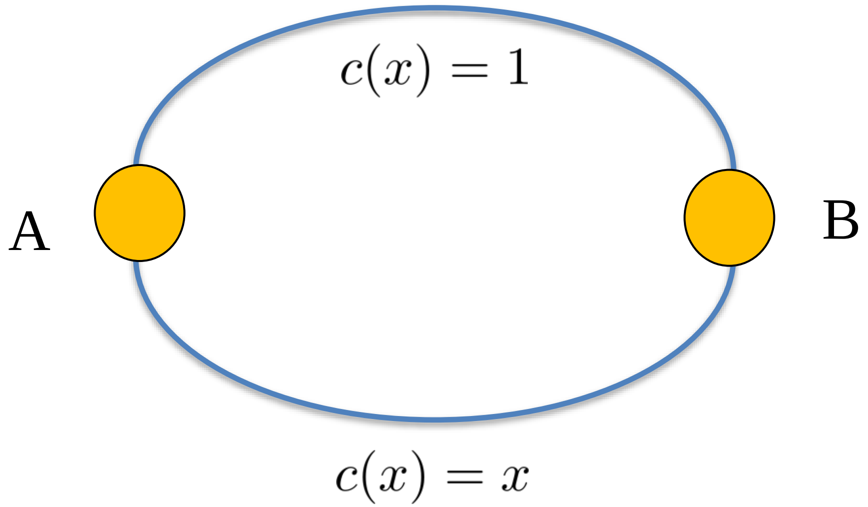
$$\text{PoA} = \frac{\text{performance of worst case NE}}{\text{optimal performance if agents do not decide on their own}}$$

Price of Anarchy (Koutsoupias, Papadimitriou 99').

4/3!!

Non-atomic selfish routing

Example: Simpler example. **Pigou network.**



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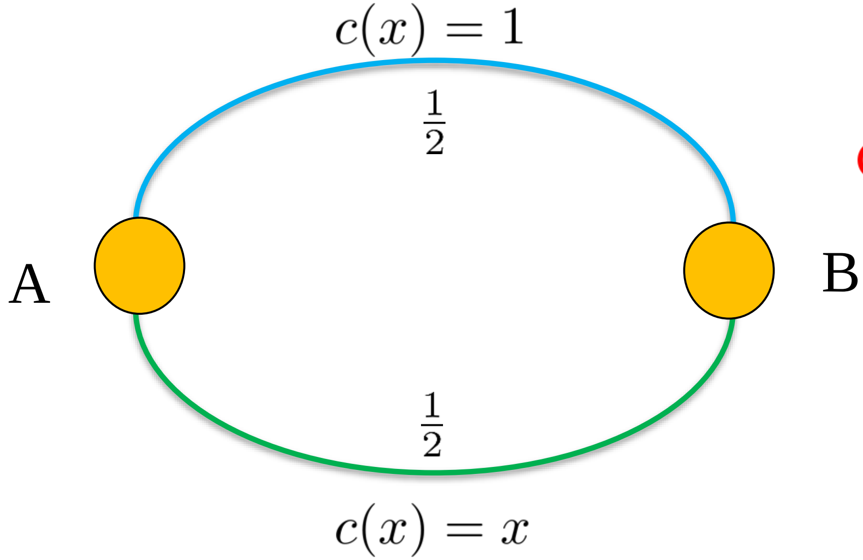

Cost $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$

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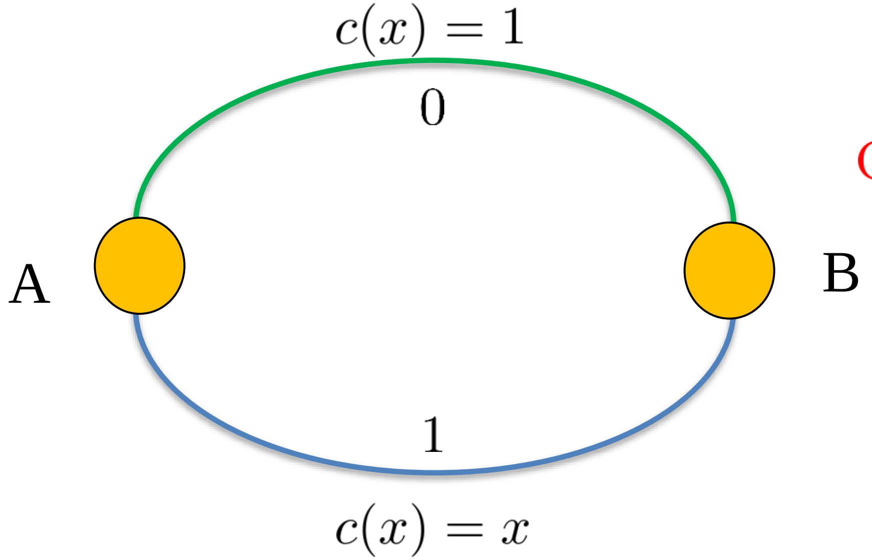

Cost 1 and $\mathbf{PoA} = \frac{4}{3}$

Non-atomic selfish routing

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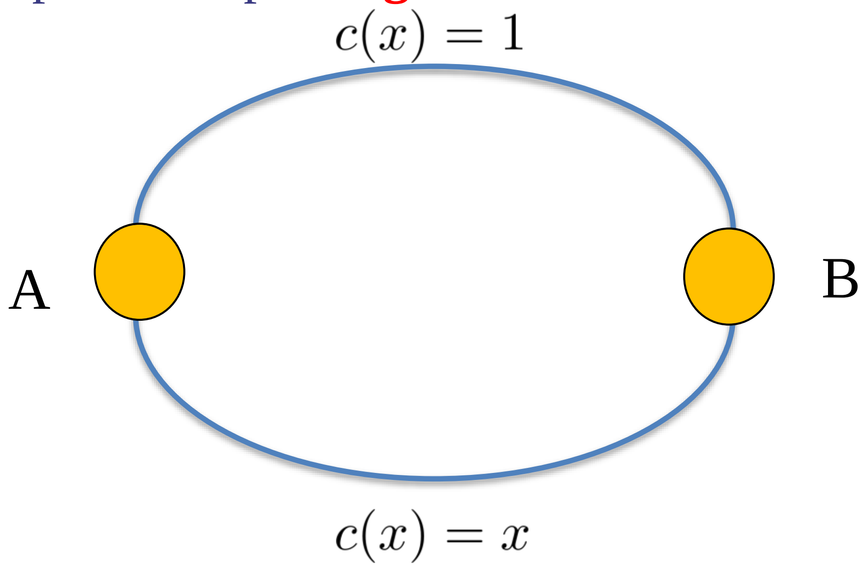

A non-atomic selfish routing game is defined by:

- Graph $G(V, E)$.
- Source destination pairs $(s_1, t_1), ..., (s_k, t_k)$.
- r_i traffic from $s_i \to t_i$.
- $c_e(.) \geq 0$ cost function of edge e, continuous and non-decreasing.
- Flow is an equilibrium if all traffic is routed on cheapest paths.

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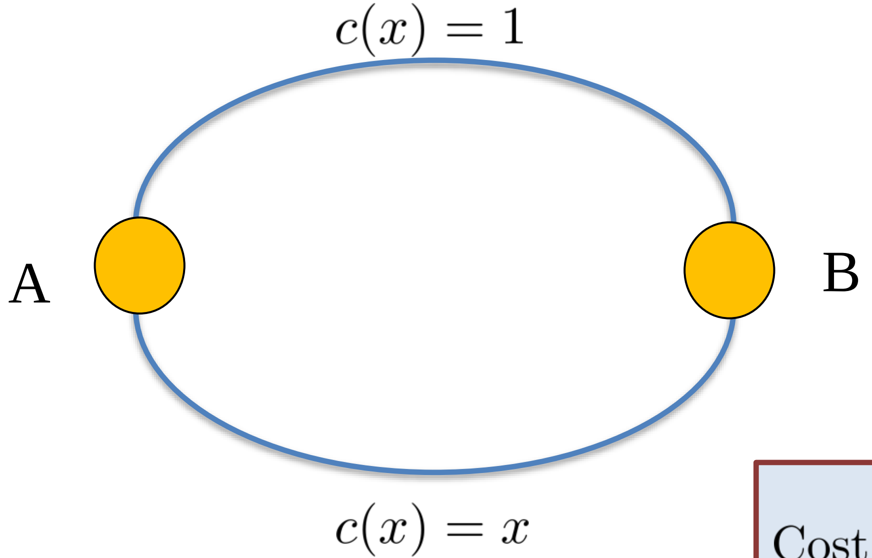

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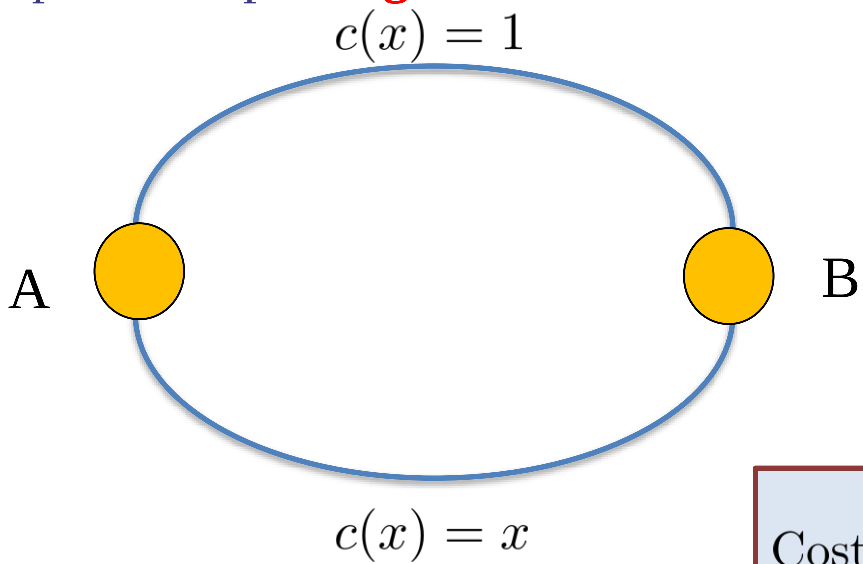

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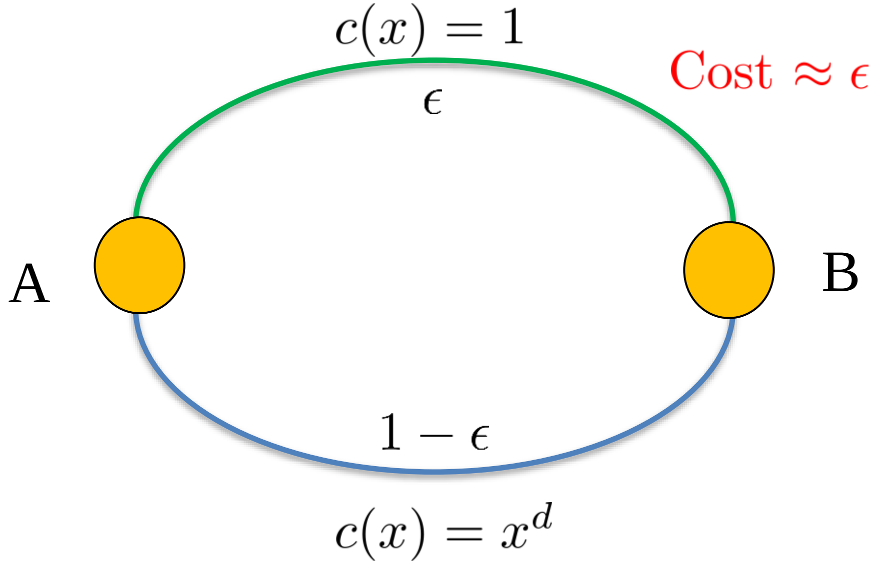
Remark: Equilibrium flow exists and is unique!

Non-atomic selfish routing

A bad Example. **Pigou network with large degree *d*.**



Cost $\approx \epsilon$ and $\mathbf{PoA} = \frac{1}{\epsilon}$ for d large.

Non-atomic selfish routing

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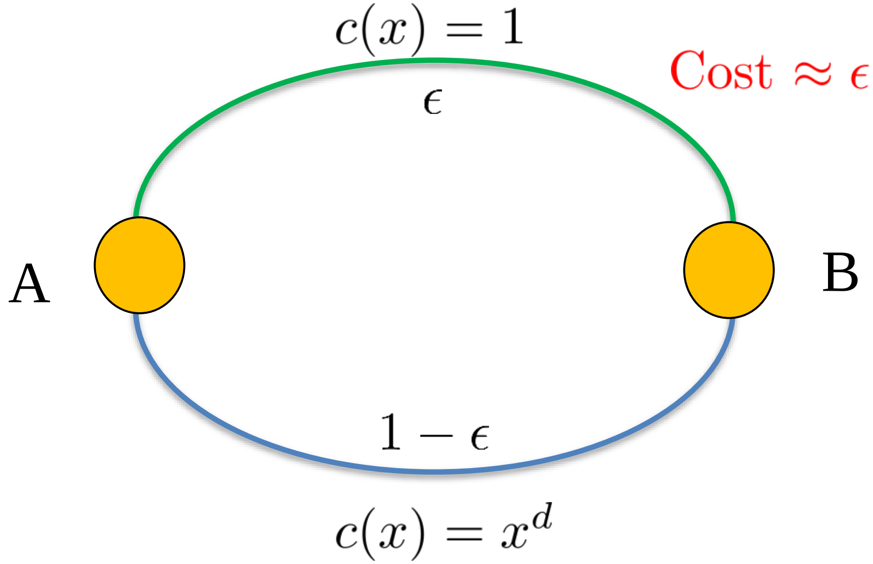

Cost $\approx \epsilon$ and $\mathbf{PoA} = \frac{1}{\epsilon}$ for d large.

Questions:

1. When is PoA small (bounded)?
2. Can we find bounds on PoA for specific classes of cost functions?

Price of Anarchy in Non-atomic selfish routing with *Linear* costs

Definition (Linear costs)**.** *Linear costs are of the form* $c_e(x) = a_e \cdot x + b_e$.

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Observe that

f^* equilibrium flow $\Rightarrow$ if $f_p^* > 0$ then $c_p(f^*) \leq c_{p'}(f^*)$ for all paths p'.

Price of Anarchy in Non-atomic selfish routing with *Linear* costs

Proof cont. Therefore all paths p so that $f_p^* > 0$ have same cost say L.

Hence $\sum_p f_p^* c_p(f^*) = L \cdot F$ where $F = \sum_p f_p^*$ is the total flow.

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Since $c_p(f^*) \geq L$ we conlude

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Combining the above

$$\sum_e f_e c_e(f^*) = \sum_p f_p c_p(f^*) \geq L \cdot F = \sum_p f_p^* c_p(f^*) = \sum_e f_e^* c_e(f^*)$$

Price of Anarchy in Non-atomic selfish routing with *Linear* costs

$$\sum_e f_e c_e(f^*) \geq \sum_e f_e^* c_e(f^*).$$

Proof cont. We get that

$$\sum_e f_e^* c_e(f^*) \leq \sum_e f_e c_e(f) + \sum_e f_e(c_e(f^*) - c_e(f))$$

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We also have that

$$\sum_e f_e(c_e(f^*) - c_e(f)) \leq \frac{1}{4} \sum_e f_e^* c_e(f^*)$$

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Since $xy - y^2 \leq \frac{x^2}{4} \Rightarrow$ LHS $\leq$ RHS.

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Or equivalently

$$\sum_e f_e^* c_e(f^*) \leq \frac{4}{3} \sum_e f_e c_e(f).$$

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Theorem (Roughgarden 02′, PoA for polynomial costs)**.** *For every network with polynomial costs with degree d:*

$$\textit{cost of Nash flow} \leq \Theta\left(\frac{d}{\log d}\right) \cdot \textit{cost of optimal flow}.$$

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HW2

Price of Anarchy in Congestion Games

Theorem (Christodoulou-Koutsoupias, PoA for linear costs)**.** *For every congestion game with linear costs:*

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$$\sum_{e \in P_i} c_e(l_e^*) \leq \sum_{e \in P_i \cap \tilde{P}_i} c_e(l_e^*) + \sum_{e \in \tilde{P}_i \setminus P_i} c_e(l_e^* + 1)$$

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&\leq \sum_{e \in P_i \cap \tilde{P}_i} c_e(l_e^* + 1) + \sum_{e \in \tilde{P}_i \setminus P_i} c_e(l_e^* + 1) \\
&= \sum_{e \in \tilde{P}_i} c_e(l_e^* + 1).
\end{aligned}$$

Price of Anarchy in Congestion Games

$$\sum_{e \in P_i} c_e(l_e^*) \leq \sum_{e \in \tilde{P}_i} c_e(l_e^* + 1).$$

Proof cont. Consider any configuration $\tilde{l}$, where each agent j uses path $\tilde{P}_j$. Summing for all agents i

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$$= \sum_e \tilde{l}_e c_e(l_e^* + 1).$$

$$= \sum_e a_e \tilde{l}_e (l_e^* + 1) + b_e \tilde{l}_e.$$

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$$=$$

$$= \sum_e a_e \tilde{l}_e (l_e^* + 1) + b_e \tilde{l}_e.$$

$$\leq \sum_e a_e \left(\frac{5}{3} \tilde{l}_e^2 + \frac{1}{3} {l_e^*}^2 \right) + b_e \tilde{l}_e.$$

Since $y(z+1) \leq \frac{5}{3} y^2 + \frac{1}{3} z^2$ for naturals y, z

HW2

Price of Anarchy in Congestion Games

$$\sum_{i\in[n]}\sum_{e\in P_i} c_e(l_e^*) \le \sum_e a_e\left(\frac{5}{3}\tilde{l}_e^2 + \frac{1}{3}l_e^{*\,2}\right) + b_e\tilde{l}_e$$

Proof cont. Observe that

$$\frac{5}{3}C(\tilde{l}) = \frac{5}{3}\sum_{i\in[n]}\sum_{e\in\tilde{P}_i} c_e(\tilde{l}_e) = \sum_e \frac{5}{3}a_e\tilde{l}_e^2 + \frac{5}{3}b_e\tilde{l}_e \ge \sum_e \frac{5}{3}a_e\tilde{l}_e^2 + b_e\tilde{l}_e$$

Price of Anarchy in Congestion Games

$$\sum_{i\in[n]}\sum_{e\in P_i} c_e(l_e^*) \le \sum_e a_e\left(\frac{5}{3}\tilde{l}_e^2+\frac{1}{3}{l_e^*}^2\right)+b_e\tilde{l}_e$$

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Therefore

$$C(l^*)\le\frac{5}{3}C(\tilde{l})+\frac{1}{3}\sum_e a_e{l_e^*}^2$$

Price of Anarchy in Congestion Games

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$$\le \frac{5}{3}C(\tilde{l}) + \frac{1}{3}C(l^*)$$

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Proof cont. Ob

$$C(l^*) \le \frac{5}{2}C(\tilde{l}).$$

$$\frac{5}{3}C(\tilde{l}) = \quad \ldots \quad a_e\tilde{l}_e^2 + b_e\tilde{l}_e$$

Therefore

$$C(l^*) \le \frac{5}{3}C(\tilde{l}) + \frac{1}{3}\sum_e a_e {l_e^*}^2$$

$$\le \frac{5}{3}C(\tilde{l}) + \frac{1}{3}C(l^*)$$

Remark:

1. The above bound is tight!
2. For polynomial cost functions the PoA is exponential in *d*.

Price of Anarchy and Balls & Bins

Definition (Balls and Bins)**.** *Consider*

- *set of n balls and n bins $\{e_1, ..., e_n\}$.*
- *Each ball i chooses a bin j and pays the load of the bin j.*
- *Define social cost the* ***maximum load****.*
- *What is PoA? Is it $\frac{5}{2}$?*

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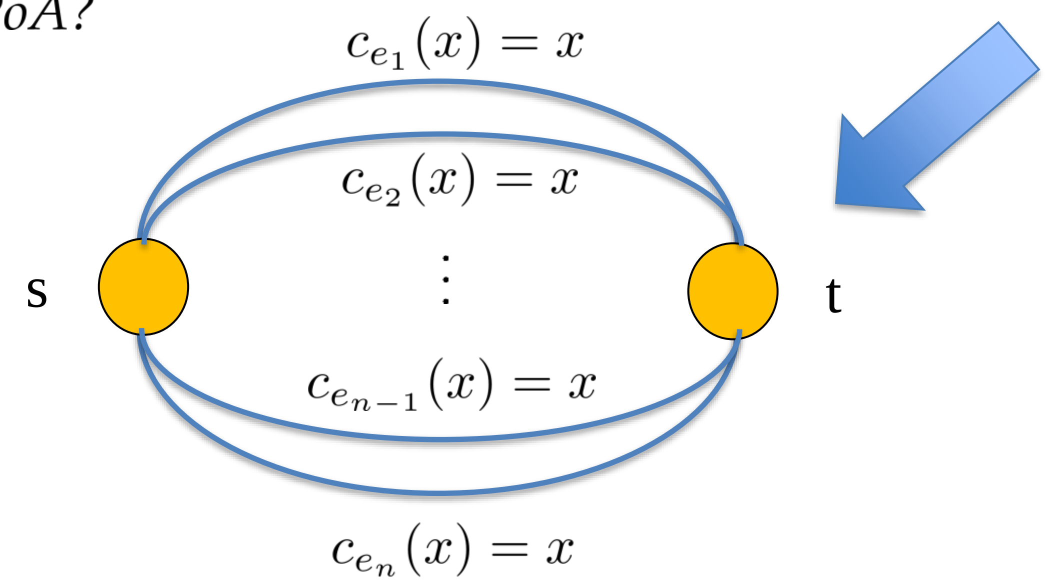
Congestion game!



Price of Anarchy and Balls & Bins

Theorem (Koutsoupias-Papadimitriou, PoA for balls & bins)**.** *The PoA is*

$$\Omega\left(\frac{\ln n}{\ln \ln n}\right).$$

Proof. We will use second moment method.

- Set every ball in a different bin. Hence optimal social cost is 1.
- Uniform $\left(\frac{1}{n}, ..., \frac{1}{n}\right)$ is a Nash Equilibrium (symmetry).

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Claim 1: Bin i has at least $k \ll n$ balls with probability at least:

$$\binom{n}{k}\frac{1}{n^k}\left(1-\frac{1}{n}\right)^{n-k} \geq \frac{1}{n^k}\left(\frac{n}{k}\right)^k\frac{1}{e} = \frac{1}{ek^k}.$$

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In general (HW2):

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Proof cont. Choosing $k = \frac{\ln n}{3 \ln \ln n}$ we have $k^k \le (\ln n)^k = (\ln n)^{\frac{\ln n}{3 \ln \ln n}} = n^{1/3}$.

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thus $Pr[X = 0] \leq Pr[|X - E[X]| \geq E[X]] \leq \frac{Var[X]}{E^2[X]}$.

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Proof cont. $Pr[X = 0] \le \frac{Var[X]}{E^2[X]}$.

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Therefore

$$Pr[X \geq 1] = 1 - Pr[X = 0] \geq 1 - \frac{n^{-1/3}}{e^2} \to 1.$$

Congestion Games

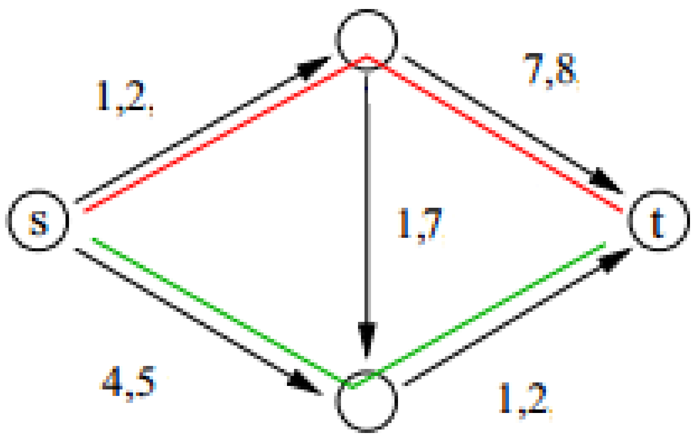
A congestion game is defined by:

- n set of players.
- E set of edges/facilities/ bins.
- $S_i \subset 2^E$ the set of strategies of player i.
- $c_e : \{1, ..., n\} \to \mathbb{R}^+$ cost function of edge e.

For any $s = (s_1, ..., s_n)$

- $l_e(s)$ number of players (load) that use edge e.
- $c_i(s) = \sum_{e \in s_i} c_e(l_e)$ the cost function of player i.

Congestion Games



For this game:

$n = \{1, 2\}$ (red, green)
E are the edges of the network.
S_i is all $s - t$ paths.
c_e on edges.

Remark: Defined by Rosenthal in 1973. Capture atomic routing **games!**